MTH 166

Lecture-9

Solution of 2nd Order Homogeneous LDE with Constant Coefficients-I

Topic:

Solution of 2^{nd} order Homogeneous LDE with Constant coefficients-I

Learning Outcomes:

1. How to apply differential operator for solving 2^{nd} order Homogeneous LDE.

2. How to write solution when roots are real and unequal.

3. How to write solution when roots are real and equal.

4. How to write solution when roots are complex conjugates.

Solution of 2nd order homogeneous LDE with constant coefficients:

Let us consider 2nd order homogeneous LDE with constant coefficients as:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

or

$$ay'' + by' + cy = 0 \qquad (1)$$

Let $\boldsymbol{\frac{d}{dx} \equiv D}$ be Differential operator (An algebraic operator like $+, -, \times, \div$)

Equation (1) becomes:

$$aD^2y + bDy + cy = 0$$

Symbolic Form (S.F.): $(aD^2 + bD + c)y = 0$

Auxiliary Equ. (A.E.): $(aD^2 + bD + c) = 0$

$(aD^2 + bD + c) = 0$

$\Rightarrow D = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = m_1, m_2$ (Say)

<u>Case 1:</u> When roots are real and unequal (distinct) i.e. $m_1 \neq m_2$

Solution: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

<u>Case 2:</u> When roots are real and equal i.e. $m_1 = m_2$

Solution: $y = (c_1 + c_2 x) e^{m_1 x}$

<u>Case 3:</u> When roots are complex conjugates i.e. $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$

Solution: $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Find the general solution of the following differential equations:

Problem 1. $y''-4y=0$

Solution: The given equation is:

$$y''-4y=0 \tag{1}$$

S.F. : $(D^2-4)y=0$ where $D\equiv\frac{d}{dx}$

A.E. : $(D^2-4)=0$ $\Rightarrow D^2=4$ $\Rightarrow D=\pm 2$ (real and unequal roots)

Let $m_1=2$ and $m_2=-2$

$\therefore$ General Solution of equation (1) is given by:

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

$\Rightarrow y=c_1e^{2x}+c_2e^{-2x}$ **Answer.**

Problem 2. $y'' - 4y' - 12y = 0$

Solution: The given equation is:

$$y'' - 4y' - 12y = 0 \tag{1}$$

S.F. : $(D^2 - 4D - 12)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 - 4D - 12) = 0 \quad \Rightarrow (D-6)(D+2) = 0$

$\Rightarrow D = 6, -2$ (real and unequal roots)

Let $m_1 = 6$ and $m_2 = -2$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{6x} + c_2 e^{-2x}$ **Answer.**

Problem 3. $y'' + 4y' + y = 0$

Solution: The given equation is:

$$y'' + 4y' + y = 0 \qquad (1)$$

S.F. : $(D^2 + 4D + 1)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 + 4D + 1) = 0$ $\quad \Rightarrow D = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2} = \frac{2(-1 \pm \sqrt{3})}{2}$

$\Rightarrow D = -1 + \sqrt{3}, -1 - \sqrt{3}$ (real and unequal roots)

Let $m_1 = -1 + \sqrt{3}$ and $m_2 = -1 - \sqrt{3}$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{(-1+\sqrt{3})x} + c_2 e^{(-1-\sqrt{3})x}$ **Answer.**

Problem 4. $y''+2y'+y=0$

Solution: The given equation is:

$$y''+2y'+y=0 \tag{1}$$

S.F. : $(D^2+2D+1)y=0$ where $D\equiv\frac{d}{dx}$

A.E. : $(D^2+2D+1)=0$ $\Rightarrow (D+1)^2=0$

$\Rightarrow D=-1,-1$ (real and equal roots)

Let $m_1=-1$ and $m_2=-1$

$\therefore$ General Solution of equation (1) is given by:

$y=(c_1+c_2x)e^{m_1x}$

$\Rightarrow y=(c_1+c_2x)e^{-1x}$**Answer.**

Problem 5. $9y'' - 12y' + 4y = 0$

Solution: The given equation is:

$$9y'' - 12y' + 4y = 0 \qquad (1)$$

S.F. : $(9D^2 - 12D + 4)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(9D^2 - 12D + 4) = 0 \qquad \Rightarrow (3D - 2)^2 = 0$

$\Rightarrow D = \frac{2}{3}, \frac{2}{3}$ (real and equal roots)

Let $m_1 = \frac{2}{3}$ and $m_2 = \frac{2}{3}$

$\therefore$ General Solution of equation (1) is given by:

$y = (c_1 + c_2 x)e^{m_1 x}$

$\Rightarrow y = (c_1 + c_2 x)e^{\frac{2}{3}x}$ **Answer.**

Problem 6. $4y'' + 4y' + 1y = 0$

Solution: The given equation is:

$$4y'' + 4y' + 1y = 0 \tag{1}$$

S.F. : $(4D^2 + 4D + 1)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(4D^2 + D + 1) = 0 \quad \Rightarrow (2D + 1)^2 = 0$

$\Rightarrow D = -\frac{1}{2}, -\frac{1}{2}$ (real and equal roots)

Let $m_1 = -\frac{1}{2}$ and $m_2 = -\frac{1}{2}$

$\therefore$ General Solution of equation (1) is given by:

$y = (c_1 + c_2 x)e^{m_1 x}$

$\Rightarrow y = (c_1 + c_2 x)e^{-\frac{1}{2}x}$ **Answer.**

Problem 7. $y'' + 25y = 0$

Solution: The given equation is:

$$y'' + 25y = 0 \qquad (1)$$

S.F. : $(D^2 + 25)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 + 25) = 0 \quad \Rightarrow D^2 = -25 \quad \Rightarrow D = \pm 5i$ (Complex conjugate roots)

Let $m_1 = 0 + 5i$ and $m_2 = 0 - 5i$ $\qquad (\alpha \pm i\beta)$

$\therefore$ General Solution of equation (1) is given by:

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$\Rightarrow y = e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$$

$$\Rightarrow y = (c_1 \cos 5x + c_2 \sin 5x)$$ **Answer.**

Problem 8. $y'' + 4y' + 5y = 0$

Solution: The given equation is:

$$y'' + 4y' + 5y = 0 \tag{1}$$

S.F. : $(D^2 + 4D + 5)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 + 4D + 5) = 0 \quad \Rightarrow D = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

Let $m_1 = -2 + 1i$ and $m_2 = -2 - 1i$ $\quad (Complex\ roots: \alpha \pm i\beta)$

$\therefore$ General Solution of equation (1) is given by:

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\Rightarrow y = e^{-2x}(c_1 \cos 1x + c_2 \sin 1x)$

$\Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x)$ **Answer.**

Problem 9. $y'' - 2y' + 2y = 0$

Solution: The given equation is:

$$y'' - 2y' + 2y = 0 \tag{1}$$

S.F. : $(D^2 - 2D + 2)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 - 2D + 2) = 0 \quad \Rightarrow D = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

Let $m_1 = 1 + 1i$ and $m_2 = 1 - 1i$ $\qquad (Complex\ roots: \alpha \pm i\beta)$

$\therefore$ General Solution of equation (1) is given by:

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\Rightarrow y = e^{1x}(c_1 \cos 1x + c_2 \sin 1x)$

$\Rightarrow y = e^{x}(c_1 \cos x + c_2 \sin x)$ **Answer.**